GENERALIZED DERIVATIVES

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0. Introduction. Generalized differential operators are ones which agree with differential operators when applied to sufficiently smooth functions but have special symmetry properties which allow them to be defined on less smooth functions. Such operators were used by Cantor [4] in his proof of the uniqueness of representation by trigonometric series and have been an integral part of all extensions of Cantor's theorem to higher dimensions.

In this article we introduce some generalized mixed partial derivative operators and a generalized biharmonic operator. We give theorems describing properties of the solutions of the related homogeneous equations and describe their connection with problems in the uniqueness of multiple trigonometric series.

1. The mixed partials. For functions $f \in C^2(\mathbf{R}^2)$, we know that $u_{xy} = 0$ for all $(x, y) \in \mathbf{R}^2$ implies that $u(x, y) = A(x) + B(y)$ 0 for all (x, y). We are interested in weakening the limiting process in taking the mixed partials and seeing if the splitting of the function u into a sum of functions of one variable still holds. We define two such limiting processes. The first is a non-centered limiting process and the second is a symmetric one.

DEFINITION 1.1. Let

$$\Delta u(x, y; h, k) = u(x+h, y+k) - u(x+h, y) - u(x, y+k) + u(x, y)$$

We then define the operator:

$$D^2_{xy} u = Du(x, y) = \lim_{\substack{h,k \to 0 \\ h,k \neq 0}} \frac{\Delta u(x, y; h, k)}{hk}$$

DEFINITION 1.2. Let

$$\Delta_s u(x, y; h, k) = u(x+h, y+k) - u(x+h, y-k) - u(x-h, y+k) + u(x-h, y-k).$$

We define:

$$D^s u(x, y) = \lim_{\substack{h,k \to 0 \\ h,k \neq 0}} \frac{\Delta_s u(x, y; h, k)}{4hk}.$$

If $f \in C^2(\mathbf{R}^2)$, then both of these agree with the mixed partial derivative. That is $f_{xy}(x, y) = Df(x, y) = D^s f(x, y)$ for all (x, y). However, these need not always coincide. Our concern in this paper is to whether one can still conclude that $u(x, y) = A(x) + B(y)$ if either $Du \equiv 0$ or $D^s u \equiv 0$. A theorem due to Bögel [3] says that if the unsymmetric generalized mixed partial $Du(x, y) = 0$ for all $(x, y) \in \mathbf{R}^2$

then $u(x,y) = A(x) + B(y)$. A simple proof of this can be found in [1]. This theorem requires no regularity assumptions on u of any kind, not even measurability.

One then would like to know if the same result will hold for the symmetrically defined generalized mixed partial. The following example shows that the two operators are different.

Example. The function $S(x,y) = \text{sgn}\,(|y| - |x|)$ is easily seen to satisfy $D^s S \equiv 0$ but doesn't decompose to a sum of a function of x and a function of y. (See [1] for details.)

It is clear from the example function that one needs additional assumptions on u to have $D^s u \equiv 0$ imply $u(x,y) = A(x) + B(y)$. We can now state our result:

THEOREM I. *If u is continuous in $\mathbf{R}^2$ and $D^s u \equiv 0$ then $u(x,y) = A(x) + B(y)$.*

Proof. In this proof a box will be a nondegenerate closed rectangle with sides parallel to the axes. Let B be a box with vertices (x,y), $(x+h,y)$, $(x+h,y+k)$, $(x,y+k)$ where $h > 0, k > 0$ and define $\Delta B = \Delta u(x,y;h,k)$. It will be of great importance in the proof that ΔB satisfies the finite additivity property

$$\Delta B = \sum \Delta B' \tag{1.3}$$

whenever $B = \cup B'$ where the $B's$ are boxes with mutually disjoint interiors.

To prove theorem I it then suffices to prove $\Delta B = 0$ for all rectangles B in $\mathbf{R}^2$ since if that is the case $u(x,y) = u(x,0) + [u(0,y) - u(0,0)]$ which is the splitting of u into functions of x and of y. We will show,

LEMMA 1.4. *u continuous and $D^s u \geq c > 0$ for all (x,y) implies $\Delta B \geq 0$ for all boxes B in $\mathbf{R}^2$.*

Remark. This will prove our theorem since if $D^s u = 0$ for all (x,y) then $D^s[u(x,y) + \epsilon xy] = \epsilon > 0$ which implies $\Delta B \geq 0$. On the other hand $D^s[-u(x,y) + \epsilon xy] = \epsilon > 0$ which implies $\Delta B \leq 0$. Hence $\Delta B = 0$ which as we have just pointed out proves the theorem.

We now prove lemma 1.4. Let S be the set of all open subsets O of $\mathbf{R}^2$ which are dense and satisfy

$$\Delta B \geq 0 \text{ for all boxes } B \subset O. \tag{1.5}$$

We will show that S is non empty and that the union of all the subsets of S is itself in S and is all of $\mathbf{R}^2$.

Let $A_n = \{(x,y) : \Delta_s u(x,y) \geq 0 \text{ for all } (h,k) \in (0,\frac{1}{n}] \times (0,\frac{1}{n}]\}$. The $A_n's$ are closed, $A_n \subset A_{n+1}$ for all n and $\cup_{n=1}^{\infty} A_n = \mathbf{R}^2$. Let A_n° denote the interior of A_n, ∂A_n the boundary of A_n and $A = \cup_{n=1}^{\infty} A_n^{\circ}$. We show that $A \in S$.

A is clearly open. Furthermore A is dense since $\mathbf{R}^2 = A \cup (\cup \partial A_n)$ implies that $A \supset (\cup \partial A_n)^c$ and $(\cup \partial A_n)^c$ is dense in $\mathbf{R}^2$ by the Baire Category Theorem. Finally, since $A_n^{\circ} \subset A_{n+1}^{\circ}$ and any box $B \subset A$ is compact, $B \subset A_N^{\circ}$ for some N. Subdivide B

into the union of subboxes B' with non-overlapping interiors and with sides of length less than $1/N$. Then $\Delta B' \geq 0$ and by finite additivity, (1.3), $\Delta B = \sum \Delta B' \geq 0$. Thus $A \in S$ and hence $S \neq \emptyset$.

Let $T = \cup_{O \in S} O$. Clearly T is open and dense in $\mathbf{R}^2$. Property (1.5) follows from a compactness argument and so T is also in S. It remains to show that $T = \mathbf{R}^2$.

Let $C = \mathbf{R}^2 \backslash T$. Assume that $C \neq \emptyset$. Then C is a complete metric space and $C = \cup_{n=1}^{\infty} (A_n \cap C)$ The sets $A_n \cap C$ are closed in C and so by the Baire Category Theorem there is an open set $O \subset \mathbf{R}^2$ with $\emptyset \neq O \cap C \subset A_n \cap C$. One can then find a box D with side lengths less than $1/n$, $D \subset O$ and $D^o \cap C \neq \emptyset$. We will show that $D^o \cup T \in S$ thus contradicting the maximality of T.

Clearly $D^o \cup T$ is open and dense. It remains to show that for any box $B \subset D^o \cup T$, $\Delta B \geq 0$. For B any such box, B can be written as the union of $B \cap D$ with at most four boxes which are entirely contained in T. Furthermore the union is disjoint except for some of the edges so that by additivity we need only show that $\Delta D \cap B \geq 0$.

We now let B denote $D \cap B$. It is clear that if $B' \subset B$ is a subbox and $B' \cap C = \emptyset$ then $\Delta B' \geq 0$. Furthermore, if B' is a subbox of B and is centered at a point $c \in C$, then $\Delta B' \geq 0$ because the sidelength of B' is less than $1/n$ and $c \in A_n$. We are then done once we prove the following lemma:

Lemma 1.6. *Let B be a box which meets the closed nowhere dense set C such that for every box $B' \subset B$*

$$B' \text{ disjoint from } C \text{ implies } \Delta B' \geq 0 \tag{1.7}$$

and

$$\text{center of } B' \text{ in } C \text{ implies } \Delta B' \geq 0, \tag{1.8}$$

then $\Delta B \geq 0$.

Proof. For ease of calculation we assume that B is centered at (0,0) and has first quadrant vertex (a, b). Let Y be the y-axis. We distinguish three cases.

Case 1: $Y \cap B \cap C = \emptyset$

For $0 < c \leq a$, let $N(c)$ be the box with vertices $(0, -b)$, $(c, -b)$, (c, b), and $(0, b)$. Since $Y \cap B$ and C are disjoint closed sets, they are separated by some positive distance. Thus, for some $c' > 0$, $N(c') \cap C = \emptyset$, and, for any $B' \subset N(c')$ we have $\Delta_s B' \geq 0$. Let $R = \{c | 0 < c \leq a \text{ and } \Delta B' \geq 0 \text{ for any } B' \subset N(c) \text{ with left edge in } Y\}$ and let $\tau = \sup\ R$. By the selection of c' above, $R \neq \emptyset$, and the continuity of f implies that $\tau \in R$. Suppose that $\tau < a$ and let $t := \min\{\tau, (a - \tau)/2\}$. We show that this gives $\tau + t \in R$, contradicting the choice of τ. It suffices to show $\Delta N(\tau + t) \geq 0$, since the same argument will apply to any box contained in $N(\tau + t)$ with left edge contained in Y.

Let V_1 be the box with opposite corners $(\tau, -b)$ and $(\tau + \frac{t}{2}, b)$. If $V_1^o \cap C \neq \emptyset$, pick $(x_1, y_1) \in V_1 \cap C$ closest to the horizontal bisector of V_1. Let B_1 be the largest box centered at (x_1, y_1) that is contained in $N(\tau+t)$ and let B_1' be the box that has the same right edge as B_1, and has left edge in Y. By (1.3) $\Delta B_1' \geq 0$ since $\Delta B_1 \geq 0$ and the left edge of B_1 is in $N(\tau)$. Note that the right edge of B_1 is contained in the right edge of $N(\tau+t)$. Now suppose V_k, (x_k, y_k), B_k, and B_k', $k = 1, ..., n-1$ have been selected. We define the box $V_n = V_{n-1} \backslash B_{n-1}$. If $V_n^o \cap C \neq \emptyset$, pick $(x_n, y_n) \in V_n \cap C$ closest to the horizontal bisector of V_n. Let B_n be the largest box centered at (x_n, y_n) that is contained in the closure of $N(\tau+t) \backslash \cup_{k=1}^n B_k$ and let B_n' be the box that has the same right edge as B_n, and has left edge in Y. If some $V_n^o \cap C = \emptyset$, we stop and obtain a finite sequence of boxes $\{B_n'\}$. Otherwise, the sequence is infinite. If $\cup B_n' = N(\tau+t)$, then, by the continuity of f, we are done. Otherwise, $N(\tau+t)$ is the union of at most four nonoverlapping boxes O_1, U_1, L_1, and R_1, of the following form. We have O_1 and U_1 as the closures of unions of adjacent elements of $\{B_n'\}$, L_1 has left edge in Y and right edge in the vertical line $x = \tau + \frac{t}{2}$ and R_1 has left edge equal to the right edge of L_1 and right edge in the right edge of $N(\gamma+t)$. (O is for "over", U for "under", L for "left", and R for "right".) Note that ΔO_1, ΔU_1, and ΔL_1 are nonnegative and $\Delta B' \geq 0$ for any box $B' \subset L_1$ with left edge in Y. If $\Delta R_1 \geq 0$ we are done by additivity.

Otherwise we iterate the process described in the preceding paragraph to obtain a nested sequence of boxes $\{R_n\}$, each R_n having right edge contained in the right edge of $N(\tau+t)$ as follows: having constructed R_{n-1}, perform the process inside the box $L_{n-1} \cup R_{n-1}$ with V_1 chosen to be the left half of R_{n-1}. If $\Delta R_n \geq 0$, we are done by additivity. If for each $n = 1, 2, ..., \Delta R_n < 0$; then $\cap R_n$ is a line segment (possibly degenerate) contained in the right edge of $N(\tau+t)$, so that $\Delta N(\tau+t) \geq 0$ by the continuity of f.

Applying the same argument to the left half of B and using additivity gives $\Delta B \geq 0$.

Case 2: $Y \cap B^o \cap C = \emptyset$.

For $0 < \epsilon < b$, let B_ϵ be the box centered at (0,0) with first quadrant vertex $(a, b-\epsilon)$. We apply Case 1 to get $\Delta B_\epsilon \geq 0$. Let $\epsilon \to 0$ and, by the continuity of f, we get $\Delta B \geq 0$.

Case 3: $Y \cap B^o \cap C \neq \emptyset$.

Let $V_1 = B$ and pick $(0, y_1) \in Y \cap V_1^o \cap C$ closest to the horizontal bisector of V_1. Let B_1 be the largest box centered at $(0, y_1)$ contained in V_1. Suppose V_k, $(0, y_k)$, and B_k, $k = 1, ..., n-1$ have been selected. We let $V_n = V_{n-1} \backslash B_{n-1}$ and pick $(0, y_n) \in Y \cap V_n^o \cap C$ closest to the horizontal bisector of V_n. Let B_n be the largest box centered at $(0, y_n)$ and contained in V_n. This generates the sequence $\{B_n\}$, which is finite if some $Y \cap V_n^o \cap C = \emptyset$. If $\cup B_n = B$, then $\Delta B \geq 0$ by the continuity of f. Otherwise, $B' = B \backslash \cup B_n$ is a box to which we apply Case 1 or 2. Again, the continuity of f, along with the fact that $\Delta B' \geq 0$, gives $\Delta B \geq 0$.

2. A generalized biharmonic operator. We now study a generalized version of the biharmonic operator Δ^2. To avoid any confusion we point out that in this section we are using the symbol Δ to denote the Laplace operator as opposed to denoting a second difference operator as it did in §1.

For $f \in C^4(\mathbf{R}^n)$, $(n \geq 2)$, it can be shown that

$$\Delta^2 u(x_o) = (a_n/R^4)[M_R u(x_o) - u(x_o) - \frac{\Delta u(x_o)}{2n} R^2] + o(1), \tag{2.1}$$

where $a_n = \Delta^2 |x|^4 = 16n + 8n^2$,

$$M_R u(x_o) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x_o)} u(t)dt$$

the average of u over the surface of the ball $B_R(x_o)$ of radius R centered at x_o, $R > 0$, ω_n is the area of the boundary of the unit sphere in $\mathbf{R}^n$, and $\partial B_R(x_o)$ is the boundary of $B_R(x_o)$. We then define the generalized biharmonic operator by:

$$G\Delta^2 u(x) = a_n \lim_{R \to 0} (1/R^4)[M_R(x) - u(x) - \frac{\Delta u(x)}{2n} R^2]. \tag{2.2}$$

We remark that the function $x^2 \operatorname{sgn} x$ satisfies $G\Delta^2 u \equiv 0$ but obviously does not satisfy $\Delta^2 u = 0$ at the origin. Our main theorem is then:

THEOREM II. *If $G\Delta^2 u(x) = 0$ for all $x \in \mathbf{R}^n$ and $\Delta u(x)$ is continuous, then $\Delta^2 u(x) = 0$ for all $x \in \mathbf{R}^n$.*

Proof. We begin by giving the following formulae:

(2.3)
$$\frac{1}{(n-2)\omega_n} \int_{B_R(0)} [\frac{1}{|t|^{n-2}} - \frac{1}{R^{n-2}}][\Delta f(x+t) - \Delta f(x)]dt = M_R f(x) - f(x) - \frac{\Delta f(x)}{2n} R^2,$$

when $n \geq 3$, and when $n = 2$,

$$\frac{1}{2\pi} \int_{B_R(0)} [\log R - \log |t|][\Delta f(x+t) - \Delta f(x)]dt = M_R f(x) - f(x) - \frac{\Delta f(x)}{4} R^2. \tag{2.4}$$

This can be verified for $n \geq 3$ by applying Green's theorem

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

over the annulus $\Omega_\epsilon = \{\epsilon < |t| < R\}$ with $u(t) = |t|^{2-n} - R^{2-n}$ and $v(t) = f(x + t) - (\Delta f(x)|t|^2/2n)$. For $n = 2$, use the same argument with $u(t) = \log R - \log |t|$. We are led to define, for $n \geq 3$:

$$D^2 g(x) = \lim_{R \to 0} \frac{c_n}{R^4} \int_{B_R(0)} [|t|^{2-n} - R^{2-n}][g(x+t) - g(x)]dt \tag{2.5}$$

where $c_n = \frac{8n^2+16n}{(n-2)\omega_n}$, with a similar definition if $n = 2$.

We will show that $D^2g(x) = 0$ and $g(x)$ continuous will imply that g is harmonic. Hence when $g = \Delta F$, $D^2 g = 0$ and ΔF is continuous, we have F is biharmonic. For $f(x)$ a function continuous on the closed ball $\overline{B_R(x_o)}$ we let $PI(f, x_o, R)$ denote the Poisson integral of f on $B_R(x_o)$. We next establish the following lemma:

LEMMA 2.6. *If $g(x)$ is continuous in $\mathbf{R}^n$ and $D^2g(x) > 0$ for all $x \in \mathbf{R}^n$, then $g(x) \le PI(g, x_o, R)(x)$ on any ball $B_R(x_o)$.*

Proof of lemma. Assume the contrary. Then there is a ball $B_R(x_o)$ and a point $x_1 \in B_R(x_o)$ such that $g(x_1) > PI(g, x_o, R)(x_1)$.

Let $w(x) = g(x) - PI(g, x_o, R)(x)$ for $x \in \overline{B_R(x_o)}$. Then $w(x)$ is zero on $\partial B_R(x_o)$ and $w(x_1) = g(x_1) - PI(g, x_o, R)(x_1) > 0$. Hence w attains its maximum at a point c in the open ball $B_R(x_o)$.

Choose $R_1 < \frac{1}{2}\text{dist}(c, \partial B_R(x_o))$. Let h be the Newtonian potential of $PI(g, x_o, R)$ for the ball $B_{R_1}(c)$. Then inside $B_{R_1}(c)$, $\Delta h(x) = PI(g, x_o, R)(x)$. Hence,

$$
\begin{aligned}
D^2w(c) =& D^2g(c) - D^2(PI(g, x_o, R))(c)\\
=& D^2g(c) - D^2(\Delta h)(c)\\
=& D^2g(c) - G\Delta^2 h(c)\\
=& D^2g(c) - \Delta^2 h(c)\\
=& D^2g(c) - \Delta PI(g, x_o, R)(c)\\
=& D^2g(c) > 0.
\end{aligned}
$$

On the other hand,

$$
D^2w(c) = \lim_{R\to 0} \frac{1}{R^4} \int_{B_R(0)} k(t, R)[w(c+t) - w(c)]dt \le 0 \tag{2.7}
$$

since $w(c+t) - w(c) \le 0$ and $k(t, R) \ge 0$, where $k(t, R) = [(n-2)\omega_n]^{-1}[|t|^{(2-n)} - R^{(2-n)}]$ if $n \ge 3$ or $(2\pi)^{-1}[\log R - \log|t|]$ if $n = 2$. Hence the assumption that $g(x_1) > PI(g, x_o, R)(x_1)$ for some x_o, R, x_1 leads to a contradiction.

Proof of theorem. We assume that $G\Delta^2 f(x) = 0$. Then for $\epsilon > 0$,

$$
D^2[\Delta(f(x) + \epsilon|x|^4)] = G\Delta^2 f(x) + 8n(n+2)\epsilon = 8n(n+2)\epsilon > 0. \tag{2.8}
$$

This tells us that the function $\Delta(f(x) + \epsilon|x|^4)$ lies below its Poisson integral on any ball in $\mathbf{R}^n$ and for any $\epsilon > 0$. Letting $\epsilon \to 0$ we conclude that if $G\Delta^2 f(x) = 0$ for all $x \in \mathbf{R}^n$, and if $\Delta f(x)$ is continuous in $\mathbf{R}^n$, then on any ball in $\mathbf{R}^n$, Δf is less than or equal to its Poisson integral over that ball.

By considering $D^2[\Delta(-f(x) + \epsilon|x|^4)]$, the same argument shows that on any ball in $\mathbf{R}^n$, $-\Delta f$ is less than or equal to its Poisson integral over that ball. Hence the conditions $G\Delta^2 f(x) = 0$ and Δf continuous will imply that Δf is harmonic which shows $\Delta^2 f(x) = 0$.

3. Concluding remarks. Our theorems were motivated in part by trying to extend Cantor's [4] theorem on the uniqueness of multiple trigonometric series:

THEOREM III. (Cantor) *If* $\lim_{R\to\infty} \sum_{|n|\le R} c_n e^{inx} = 0$ *for all* $x \in \mathbf{R}$, *then* $c_n = 0$ *for all* $n \in Z$.

The proof of Cantor's theorem involves looking at the formal second integral $F(x) = c_o x^2/2 - \sum_{n\neq 0}(c_n/n^2)e^{inx}$. The pointwise convergence of the original series implies that the $c_n s$ go to zero (Cantor-Lebesgue) and this control of the $c_n s$ implies that $F(x)$ is continuous. Term by term application of the second derivative to the series defining F suggests that $F''(x) \equiv 0$ from which it would immediately follow that the $c_n s$ are all zero. However, the convergence of the original series is not uniform and so one cannot interchange summation and differention. Instead, Cantor used the following:

LEMMA 3.2. *If* $F(x)$ *is continuous and its Schwartz derivative*

$$DF(x) := \lim_{h\to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = 0$$

then $F(x)$ *is a line.*

Our Theorems I and II are analogs of Lemma 3.2. Our ultimate goal is to refine these results to where they can be applied to the appropriate formally integrated trigonometric series in d dimensions and thus obtain uniqueness theorems.

Shapiro proved [7] that if $c_n = o(R^{2-d})$ and $\sum c_n e^{inx}$ Abel sums to zero for all x where $c_n \in \mathbf{C}$, $n \in Z^d$ and $nx = n_1x_1 + ... + n_dx_d$, then $c_n = 0$ for all $n \in Z^d$. Roger Cooke [6] later proved that the circular convergence of a two dimensional trigonometric series for all x implies that the coefficients c_n satisfy Shapiro's hypothesis and since circular convergence implies Abel summability one combines Shapiro and Cooke's works to obtain:

THEOREM IV. *If* $\lim_{r\to\infty} \sum_{|n|\le r} c_n e^{inx} = 0$ *for all* $x \in \mathbf{R}^2$, *then* $c_n = 0$ *for all* $n \in Z^2$. *Later, estimates for the coefficients of spherically convergent trigonometric series were obtained by Bernard Connes [5] in all dimensions but they only satisfy Shapiro's requirements in dimension two.*

We can apply our theorem II in two dimensions to the series $\sum(c_m/|m|^4)e^{imx}$ where it is assumed that $\sum c_m e^{imx}$ circularly sums to zero everywhere. Using Shapiro's result that $\lim_{t\to 0}\sum(c_m/|m|^2)e^{imx-|m|t}$ is continuous, and Cooke's coefficient bounds we can obtain another (more complicated) proof of uniqueness.

For the case of unrestricted rectangular convergence Ash and Welland[2] proved:

THEOREM V. *If*

$$\lim_{\min(M,N)\to\infty} \sum_{|m|\le M} \sum_{|n|\le N} c_{mn} e^{i(mx+ny)} = 0 \qquad \text{for all } (x,y) \in \mathbf{R}^2,$$

then $c_{mn} = 0$ *for all* $(m,n) \in Z^2$.

The proof relied in a crucial way on Shapiro's results and did not generalize to higher dimensions. Our theorem I is part of a program to develop an inductive procedure to show that in d-dimensions the assumption

$$\lim_{\min(N_1,\ldots,N_d)\to\infty} \sum_{|n_1|\le N_1} \cdots \sum_{|n_d|\le N_d} c_{n_1\ldots n_d} e^{i(n_1x_1+\ldots+n_dx_d)} = 0$$

$\forall x \in \mathbf{R}^d$ implies that the formal d^{th} integral $F(x)$ obtained by termwise integration in each variable separately is a sum of functions of $d-1$ variables.

In two dimensions we have shown [1] that $D^2_{L^2,s}F(x) = 0$ ($D^2_{L^2,s}$ is a square integrably averaged version of the operator D^2_s of this paper), where F is the twice formally integrated trigonometric series first in x and then in y. Our counterexample shows that additional regularity (such as continuity) is needed to obtain $F(x,y) = A(x) + B(y)$. We have not been able to show directly that $F(x,y)$ is continuous. (Continuity of course follows from the uniqueness theorem [2] since in that case the coefficients are all zero). However we are able to show in one dimension that $c_n = o(\frac{1}{n})$ implies that $\sum c_n e^{inx} \in VMO$. Our conjecture is then that in d-dimensions if the suitably defined generalized mixed partial operator $D^d_{L^2,s}F(x) = 0$ for all $x \in \mathbf{R}^d$, and F is in some suitably defined product VMO, then F can be written as the sum of functions of $d-1$ variables.

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